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On Approximation in the L^p-Norm by Reciprocals of Polynomials

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1. INTRODUCTION

Recently the second and third authors have considered the question of approximating a real-valued continuous function f on $[-1, 1]$ by reciprocals of polynomials having real or complex coefficients. While no restrictions on f are necessary for the approximation by reciprocals of complex polynomials, it is obvious that if we limit ourselves to reciprocals of real polynomials we must assume that f does not change sign in the interval. Under this assumption it was shown in [3] that one can approximate $f(\not\equiv 0)$ by reciprocals of real polynomials at the rate $\omega(f, 1/n)$, where $\omega(f, \cdot)$ is the usual modulus of continuity of f. The purpose of this note is to improve the above estimates by replacing $\omega(f, 1/n)$ by the Ditzian–Totik modulus of continuity $\omega_\varphi(f, 1/n)$ and also to obtain estimates on the rate of approximation by reciprocals of polynomials in the L^p-norm, $1 \leqslant p < \infty$.

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Here, unfortunately, we have to assume $f \in L^{p+1}[-1, 1]$ and give the estimates in terms of $\omega_\varphi(f, 1/n)_{p+1}$. The last section is devoted to some estimates on shape-preserving approximation by reciprocals of polynomials in the various norms.

2. Approximation in $C[-1, 1]$

Let $\varphi(x) := \sqrt{1-x^2}$ and set

$$\Delta_{h\varphi} f(x) := \begin{cases} f(x + (h/2)\,\varphi(x)) - f(x - (h/2)\,\varphi(x)), & x \pm (h/2)\,\varphi(x) \in [-1, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Following Ditzian and Totik [2], define

$$\omega_\varphi(f, t) := \sup_{0 < h \leqslant t} \|\Delta_{h\varphi} f\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the sup norm over $[-1, 1]$. Then it is readily seen that, for any $f \in C[-1, 1]$,

$$\omega_\varphi(f, t) \leqslant \omega(f, t)$$

while, for instance, for $f(x) = \sqrt{1+x}$ we have

$$\omega_\varphi(f, t) = O(t) \qquad \text{and} \qquad \omega(f, t) \sim t^{1/2}.$$

Ditzian and Totik [2] proved that $\omega_\varphi(f, t)$ is equivalent to the modified Peetre kernel

$$K_\varphi(f, t) := \inf\{\|f - g\|_\infty + t\,\|\varphi g'\|_\infty + t^2\,\|g'\|_\infty\}.$$

where the infimum is taken over all g that are absolutely continuous in $[-1, 1]$ and such that $g' \in L^\infty[-1, 1]$. Our first main result is

THEOREM 1. *Let $f \in C[-1, 1]$ be nonconstant and nonnegative. Then there exists a sequence of polynomials $\{p_n\}_1^\infty$, with $p_n \in \mathscr{P}_n$, such that*

$$\left\| f - \frac{1}{p_n} \right\|_\infty \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right), \qquad n = 1, 2, \ldots. \tag{1}$$

Here and throughout this paper, C is an absolute constant independent of f and n whose value may be different from line to line and $\mathscr{P}_n$ denotes the collection of all real polynomials of degree at most n.

Remark. Obviously, a nonzero constant function f is approximable at the rate (1), while $f \equiv 0$ is not.

In the proof of Theorem 1 we shall need the following.

LEMMA 2. *Let $f \in C[-1, 1]$ and define $g \in C[-\pi, \pi]$ by $g(\theta) := f(\cos\theta)$. Let $K_n(t)$ be the Jackson kernel that satisfies*

$$\int_{-\pi}^{\pi} K_n(t)\,dt = 1, \qquad \int_{-\pi}^{\pi} |t|^k K_n(t)\,dt \sim n^{-k}, \qquad k = 1, 2, 3, 4. \tag{2}$$

Then, for $-\pi \leqslant \theta \leqslant \pi$,

$$\int_{-\pi}^{\pi} |g(\theta+t) - g(\theta)|^k K_n(t)\,dt \leqslant C\left[\omega_\varphi\left(f, \frac{1}{n}\right)\right]^k, \qquad k = 1, 2.$$

Proof. By virtue of the equivalence between $\omega_\varphi(f, t)$ and $K_\varphi(f, t)$, given $f \in C[-1, 1]$, for each $n = 1, 2, \ldots$, there exists an f_n absolutely continuous on $[-1, 1]$ such that

$$\begin{aligned} \|f - f_n\|_\infty &\leqslant C\omega_\varphi\left(f, \frac{1}{n}\right), \\ \|\varphi f_n'\|_\infty &\leqslant Cn\omega_\varphi\left(f, \frac{1}{n}\right), \\ \|f_n'\|_\infty &\leqslant Cn^2\omega_\varphi\left(f, \frac{1}{n}\right). \end{aligned} \tag{3}$$

Setting $g_n(\theta) := f_n(\cos\theta)$, then by (2) and (3) we have for $k = 1, 2$,

$$\begin{aligned} &\int_{-\pi}^{\pi} |g(\theta+t) - g(\theta)|^k K_n(t)\,dt \\ &\qquad \leqslant C\left[\omega_\varphi\left(f, \frac{1}{n}\right)\right]^k + \int_{-\pi}^{\pi} |g_n(\theta+t) - g_n(\theta)|^k K_n(t)\,dt. \end{aligned} \tag{4}$$

Now, for each u between $\cos\theta$ and $\cos(\theta + t)$ we have

$$\begin{aligned} 1 &= -\frac{2\sin(\theta + t/2)\sin(t/2)}{\cos(\theta+t) - \cos\theta} \\ &= -\frac{2\varphi(u)\sin(t/2)}{\cos(\theta+t) - \cos\theta} + \frac{2[\varphi(u) - \sin(\theta + t/2)]\sin(t/2)}{\cos(\theta+t) - \cos\theta}. \end{aligned}$$

Hence

$$\int_{-\pi}^{\pi} |g_n(\theta+t)-g_n(\theta)|^k K_n(t)\,dt$$

$$= \int_{-\pi}^{\pi} \left| \int_{\cos\theta}^{\cos(\theta+t)} f'_n(u)\,du \right|^k K_n(t)\,dt$$

$$\leqslant C \int_{-\pi}^{\pi} \left| \frac{1}{\cos(\theta+t)-\cos\theta} \int_{\cos\theta}^{\cos(\theta+t)} |f'_n(u)|\,\varphi(u)\,du \right|^k |t|^k K_n(t)\,dt$$

$$+ C \int_{-\pi}^{\pi} \left| \frac{1}{\cos(\theta+t)-\cos\theta} \int_{\cos\theta}^{\cos(\theta+t)} |f'_n(u)|\,du \right|^k |t|^{2k} K_n(t)\,dt$$

$$\leqslant C\,\|f'_n\varphi\|_\infty^k \int_{-\pi}^{\pi} |t|^k K_n(t)\,dt + C\,\|f'_n\|_\infty^k \int_{-\pi}^{\pi} |t|^{2k} K_n(t)\,dt. \tag{5}$$

Thus, from (2) and (3), we conclude that for $k=1, 2$,

$$\int_{-\pi}^{\pi} |g_n(\theta+t)-g_n(\theta)|^k K_n(t)\,dt \leqslant C\{\|f'_n\varphi\|_\infty^k n^{-k} + \|f'_n\|_\infty^k n^{-2k}\}$$

$$\leqslant C\left[\omega_\varphi\left(f, \frac{1}{n}\right)\right]^k.$$

Combining this last inequality with (4) proves Lemma 2. ∎

We now turn to the proof of Theorem 1. Although we basically follow the ideas of the proof in [3] (except that Lemma 2 provides sharper estimates), there is one major difference. In preparation for the L^p case we do not wish to use the pointwise value of f in order to get a lower estimate on the product fp_n and thereby prove that p_n does not vanish in $[-1, 1]$. In fact, in the L^p case, it makes no sense to look for such a lower estimate. Nevertheless we prove that p_n does not vanish in $[-1, 1]$.

Proof of Theorem 1. Given a nonconstant $f \in C[-1, 1]$, $f \geqslant 0$, and $\varepsilon > 0$, let $f_\varepsilon(x) := f(x) + \varepsilon$ and let $g_\varepsilon(\theta) := f_\varepsilon(\cos\theta)$, $\theta \in [-\pi, \pi]$. Then $|1/g_\varepsilon| \leqslant 1/\varepsilon$ and we can define the algebraic polynomials

$$p_n(x) := \int_{-\pi}^{\pi} \frac{1}{g_\varepsilon(\theta+t)} K_n(t)\,dt, \qquad n = 1, 2, \ldots, \tag{6}$$

where $x = \cos\theta$ and $K_n(t)$ is the Jackson kernel of Lemma 2.

By Hölder's inequality,

$$1 = \left(\int_{-\pi}^{\pi} K_n(t)\,dt\right)^2$$

$$\leqslant \int_{-\pi}^{\pi} g_\varepsilon(\theta+t)\,K_n(t)\,dt \cdot \int_{-\pi}^{\pi} \frac{1}{g_\varepsilon(\theta+t)}\,K_n(t)\,dt$$

$$= p_n(x) \cdot \int_{-\pi}^{\pi} g_\varepsilon(\theta+t)\,K_n(t)\,dt.$$

Thus p_n does not vanish in $[-1, 1]$ and

$$\frac{1}{p_n(x)} \leqslant \int_{-\pi}^{\pi} g_\varepsilon(\theta+t)\,K_n(t)\,dt. \tag{7}$$

Let $E := \{x\colon (1/p_n(x)) > f_\varepsilon(x)\}$. Then, by (7) and Lemma 2, we have for $x \in E$

$$0 < \frac{1}{p_n(x)} - f_\varepsilon(x) \leqslant \int_{-\pi}^{\pi} [g_\varepsilon(\theta+t) - g_\varepsilon(\theta)]\,K_n(t)\,dt$$

$$\leqslant C\omega_\varphi\left(f_\varepsilon, \frac{1}{n}\right) = C\omega_\varphi\left(f, \frac{1}{n}\right). \tag{8}$$

For x in the complement of E we have

$$\frac{1}{p_n(x)} \leqslant f_\varepsilon(x).$$

Hence

$$0 \leqslant f_\varepsilon(x) - \frac{1}{p_n(x)} = \int_{-\pi}^{\pi} \left[\frac{1}{g_\varepsilon(\theta+t)} - \frac{1}{g_\varepsilon(\theta)}\right] \frac{g_\varepsilon(\theta)}{p_n(\cos\theta)}\,K_n(t)\,dt$$

$$\leqslant \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta) - g_\varepsilon(\theta+t)|}{g_\varepsilon(\theta)\,g_\varepsilon(\theta+t)}\,\frac{g_\varepsilon(\theta)}{p_n(\cos\theta)}\,K_n(t)\,dt$$

$$\leqslant \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta) - g_\varepsilon(\theta+t)|}{g_\varepsilon(\theta+t)}\,g_\varepsilon(\theta)\,K_n(t)\,dt$$

$$\leqslant \int_{-\pi}^{\pi} |g_\varepsilon(\theta) - g_\varepsilon(\theta+t)|\,K_n(t)\,dt + \frac{1}{\varepsilon}\int_{-\pi}^{\pi} |g_\varepsilon(\theta) - g_\varepsilon(\theta+t)|^2\,K_n(t)\,dt,$$

where for the last inequality we used the fact that $1/g_\varepsilon \leqslant 1/\varepsilon$. By virtue of Lemma 2, we have for $x \notin E$

$$0 \leqslant f_\varepsilon(x) - \frac{1}{p_n(x)} \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right) + C\,\frac{1}{\varepsilon}\left[\omega_\varphi\left(f, \frac{1}{n}\right)\right]^2$$

(since $\omega_\varphi(f_\varepsilon, t) \equiv \omega_\varphi(f, t)$). Choosing $\varepsilon = \omega_\varphi(f, 1/n)$, which is not zero since $f \not\equiv$ const, yields

$$0 \leqslant f_\varepsilon(x) - \frac{1}{p_n(x)} \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right), \qquad \text{for } x \notin E.$$

Combining with (8) we have

$$\left\| f_\varepsilon - \frac{1}{p_n} \right\|_\infty \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right).$$

Thus

$$\begin{aligned}\left\| f - \frac{1}{p_n} \right\|_\infty &\leqslant \|f - f_\varepsilon\|_\infty + \left\| f_\varepsilon - \frac{1}{p_n} \right\|_\infty \\ &\leqslant \varepsilon + C\omega_\varphi\left(f, \frac{1}{n}\right) \\ &\leqslant C\omega_\varphi\left(f, \frac{1}{n}\right).\end{aligned}$$

This completes the proof. ▮

Remark. If we work with $C[0, 1]$ instead of $C[-1, 1]$, then φ takes the form $\varphi(x) = \sqrt{x(1-x)}$ and for x^α, $0 < \alpha < 1$, we have $\omega_\varphi(x^\alpha, t) = O(t^{2\alpha})$. Hence the error in approximating x^α, $0 < \alpha < 1$, on $[0, 1]$ by reciprocals of polynomials can be estimated by $Cn^{-2\alpha}$, where C is an absolute constant. This fact was also proved in [3] where a special construction is used. Note, however, that our present proof is valid only for $0 < \alpha < 1$, while in [3] a similar estimate is established for all $\alpha > 0$ with $C = C(\alpha)$ increasing to infinity as $\alpha \to \infty$.

3. APPROXIMATION IN $L^p[-1, 1]$

Here again we follow Ditzian and Totik [2] as we denote

$$\omega_\varphi(f, t)_p := \sup_{0 \leqslant h \leqslant t} \|\Delta_{h\varphi} f\|_p.$$

It was shown in [2] that $\omega_\varphi(f, t)_p$ is equivalent to the Peetre kernel

$$K_\varphi(f, t)_p := \inf\{\|f - g\|_p + t\,\|\varphi g'\|_p + t^2\,\|g'\|_p\},$$

where the infimum is taken over all $g \in L^p[-1, 1]$ that are absolutely continuous in $[-1, 1]$ and such that $g' \in L^p[-1, 1]$.

Our result in this case is not as satisfactory as in $C[-1, 1]$. We will prove

THEOREM 3. *Let $f \in L^{p+1}[-1, 1]$, $1 \leqslant p < \infty$, be nonconstant and nonnegative. Then there exists a sequence of polynomials $\{p_n\}_1^\infty$, with $p_n \in \mathscr{P}_n$, such that*

$$\left\| f - \frac{1}{p_n} \right\|_p \leqslant C\omega_\varphi \left(f, \frac{1}{n} \right)_{p+1}, \qquad n = 1, 2, ... \tag{9}$$

Remark. Obviously $L^{p+1}[-1, 1]$ is a proper subset of $L^p[-1, 1]$ and we have the inequality

$$\omega_\varphi(f, t)_p \leqslant \omega_\varphi(f, t)_{p+1},$$

but we are not able to replace the right-hand side of (9) by $C\omega_\varphi(f, 1/n)_p$. We do not know if this gap is indeed necessary or is due to the limitations of our method of proof.

Proof of Theorem 3. It follows from the equivalence of $\omega_\varphi(f, \cdot)_{p+1}$ and $K_\varphi(f, \cdot)_{p+1}$ that, for each n, there exists an absolutely continuous function $f_n \in L^{p+1}[-1, 1]$ such that

$$\|f - f_n\|_p \leqslant C \|f - f_n\|_{p+1} \leqslant C\omega_\varphi \left(f, \frac{1}{n} \right)_{p+1},$$

$$\|\varphi f'_n\|_{p+1} \leqslant Cn\omega_\varphi \left(f, \frac{1}{n} \right)_{p+1},$$

$$\|f'_n\|_{p+1} \leqslant Cn^2\omega_\varphi \left(f, \frac{1}{n} \right)_{p+1}. \tag{10}$$

Moreover, a close look at the proof of Ditzian and Totik [2, Sect. 3.1] reveals that f_n is nonnegative if $f \geqslant 0$. Thus it suffices to approximate f_n at the proper rate and this together with (10) will yield (9).

We proceed as in the proof of Theorem 1. Let $F_n(x) := f_n(x) + \varepsilon$ and let $g_\varepsilon(\theta) := g_{\varepsilon,n}(\theta) := F_n(\cos\theta)$, $-\pi \leqslant \theta \leqslant \pi$. Let $K_n(t)$ be a suitable Jackson kernel, i.e., such that

$$\int_{-\pi}^{\pi} K_n(t)\, dt = 1, \qquad \int_{-\pi}^{\pi} |t|^k K_n(t)\, dt \sim n^{-k}, \qquad k = 1, 2, ..., [2p+3]. \tag{11}$$

Then again $g_\varepsilon^{-1} \leqslant 1/\varepsilon$ and we can define the polynomial p_n by (6). We still have the estimate (7), although the right-hand side of (7) may be infinite for $f \in L^{p+1}[-1, 1]$. That this is not so for a differentiable f follows from (14) and (15) later in our proof.

Let

$$E_1 := \left\{x: \frac{1}{p_n(x)} > F_n(x)\right\}.$$

Then, by (7) and Minkowski's inequality,

$$\left[\int_{E_1} \left|\frac{1}{p_n(x)} - F_n(x)\right|^p dx\right]^{1/p}$$
$$\leqslant \left[\int_{E_1} \left|\int_{-\pi}^{\pi} [g_\varepsilon(\theta+t) - g_\varepsilon(\theta)]\, K_n(t)\, dt\right|^p dx\right]^{1/p}$$
$$\leqslant \int_{-\pi}^{\pi} K_n(t) \left[\int_{E_1} |g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^p\, dx\right]^{1/p} dt. \tag{12}$$

Next, for any $x \in [-1, 1]$,

$$\left|\frac{1}{p_n(x)} - F_n(x)\right| = \frac{|1 - p_n(x)\, F_n(x)|}{p_n(x)}$$
$$\leqslant \frac{1}{p_n(x)} \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|}{g_\varepsilon(\theta+t)}\, K_n(t)\, dt.$$

and so using the integral representation (6) and Hölder's inequality we get

$$\left|\frac{1}{p_n(x)} - F_n(x)\right| \leqslant \left[\frac{1}{p_n(x)} \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^p}{g_\varepsilon(\theta+t)}\, K_n(t)\, dt\right]^{1/p}.$$

Now for $x \in E_2 := [-1, 1] \setminus E_1$, we have

$$\frac{1}{p_n(x)} \leqslant F_n(x),$$

and so it follows that for $x \in E_2$

$$\left|\frac{1}{p_n(x)} - F_n(x)\right|^p \leqslant \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^p}{g_\varepsilon(\theta+t)}\, g_\varepsilon(\theta)\, K_n(t)\, dt.$$

Hence

$$\int_{E_2} \left|\frac{1}{p_n(x)} - F_n(x)\right|^p dx$$
$$\leqslant \int_{E_2} \int_{-\pi}^{\pi} |g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^p\, K_n(t)\, dt\, dx$$
$$+ \frac{1}{\varepsilon} \int_{E_2} \int_{-\pi}^{\pi} |g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^{p+1}\, K_n(t)\, dt\, dx, \tag{13}$$

where we used the inequality $g_\varepsilon^{-1} \leqslant 1/\varepsilon$.

It remains to estimate the integrals on the right-hand sides of (12) and (13). They are similar and we use the method of proof of Lemma 2 in order to estimate each of them. What we get is

$$\int_{-\pi}^{\pi} K_n(t)\left[\int_{E_1} |g_\varepsilon(\theta+t)-g_\varepsilon(\theta)|^p\,dx\right]^{1/p} dt \leqslant C\omega_\varphi\left(f,\frac{1}{n}\right)_{p+1}, \tag{14}$$

$$\int_{-\pi}^{\pi} K_n(t)\int_{E_2} |g_\varepsilon(\theta+t)-g_\varepsilon(\theta)|^q\,dx\,dt \leqslant C\left[\omega_\varphi\left(f,\frac{1}{n}\right)_{p+1}\right]^q, \tag{15}$$

for $q=p$ or $q=p+1$.

We shall only prove (15) ((14) being similar). Consider

$$\begin{aligned}
&\int_{-\pi}^{\pi} K_n(t)\int_{E_2} |g_\varepsilon(\theta+t)-g_\varepsilon(\theta)|^q\,dx\,dt \\
&\quad=\int_{-\pi}^{\pi} K_n(t)\int_{E_2}\left|\int_{\cos\theta}^{\cos(\theta+t)} F_n'(u)\,du\right|^q dx\,dt \\
&\quad\leqslant C\int_{-\pi}^{\pi}\left|\frac{1}{\cos(\theta+t)-\cos\theta}\int_{\cos\theta}^{\cos(\theta+t)} |F_n'(u)|\,\varphi(u)\,du\right|^q |t|^q K_n(t)\,dx\,dt \\
&\qquad+C\int_{-\pi}^{\pi}\left|\frac{1}{\cos(\theta+t)-\cos\theta}\int_{\cos\theta}^{\cos(\theta+t)} |F_n'(u)|\,du\right|^q |t|^{2q} K_n(t)\,dx\,dt
\end{aligned}$$

as in the proof of Lemma 2. Denote by $M_F(x)$ the Hardy maximal function of F, i.e.,

$$M_F(x) := \sup_{x\in I}\frac{1}{|I|}\left|\int_I F(s)\,ds\right|.$$

Then it follows that for $x=\cos\theta$

$$\begin{aligned}
&\int_{-\pi}^{\pi} K_n(t)\int_{E_2} |g_\varepsilon(\theta+t)-g_\varepsilon(\theta)|^q\,dx\,dt \\
&\quad\leqslant C\int_{-\pi}^{\pi} |t|^q K_n(t)\int_{E_2} |M_{|F_n'|\varphi}(x)|^q\,dx\,dt \\
&\qquad+C\int_{-\pi}^{\pi} |t|^{2q} K_n(t)\int_{E_2} |M_{|F_n'|}(x)|^q\,dx\,dt \\
&\quad\leqslant Cn^{-q}\,\|M_{|F_n'|\varphi}\|_q^q + Cn^{-2q}\,\|M_{|F_n'|}\|_q^q \\
&\quad\leqslant Cn^{-q}\,\|M_{|F_n'|\varphi}\|_{p+1}^q + Cn^{-2q}\,\|M_{|F_n'|}\|_{p+1}^q \\
&\quad\leqslant Cn^{-q}\,\|F_n'\varphi\|_{p+1}^q + Cn^{-2q}\,\|F_n'\|_{p+1}^q,
\end{aligned}$$

by virtue of the inequality (see [4, p. 58])

$$\|M_F\|_p \leqslant C_p \|F\|_p, \qquad 1 < p \leqslant \infty.$$

The proof of (15) now follows from (10).

Finally, we choose $\varepsilon = \omega_\varphi(f, 1/n)_{p+1}$. Then (12) through (15) yield

$$\left\| F_n - \frac{1}{p_n} \right\|_p \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right)_{p+1},$$

which together with (10) proves (9). ∎

4. Shape-Preserving Approximation

Returning to continuous functions we will show that a monotone increasing $f \in C[-1, 1]$ is approximable by reciprocals of monotone decreasing polynomials p_n (so that $1/p_n$ is monotone increasing) at the same rate (1). To this end we observe that Beatson [1] proved the existence of a Jackson-type kernel satisfying (2) and such that it takes increasing functions into increasing functions. Using this kernel in the proof of Theorem 1, we see that whenever f is increasing so is f_ε and hence f_ε^{-1} is decreasing. Therefore the polynomials p_n defined by (6) are decreasing. We summarize these observations in

Theorem 4. *Let $f \in C[-1, 1]$ be nonnegative and increasing. Then for each n there is a decreasing $p_n \in \mathscr{P}_n$ such that* (1) *holds.*

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